

# Quantifier elimination for approximate BK-factorization

E. Kartashova<sup>1</sup>, S. McCallum<sup>2</sup>,

<sup>1</sup> RISC, J.Kepler University, Linz, Austria

<sup>2</sup> Macquarie University, Sydney, Australia

e-mail: lena@risc.uni-linz.ac.at, scott@ics.mq.edu.au

## 1 Introduction

Factorization of linear partial differential operators (LPDOs) is a very well-studied problem and a lot of pure existence theorems are known. The only known constructive factorization algorithm - Beals-Kartashova (BK) factorization - is presented in [1]). Its comparison with Hensel descent which is sometimes regarded as constructive, is given in [2], where the idea to use BK-factorization for *approximate* factorization is also discussed. It originates in one of the most interesting features of BK-factorization: at the beginning all the first-order factors are constructed and afterwards the factorization condition(s) should be checked. This leads to the important application area - namely, numerical simulations which could be simplified substantially if instead of computation with one LPDE of order  $n$  we will be able to proceed computations with  $n$  LPDEs all of order 1. In numerical simulations it is not necessary to fulfill factorization conditions exactly but with some given accuracy, which we call approximate factorization.

The idea of the present paper is to look into the feasibility of solving problems of this kind using *quantifier elimination by cylindrical algebraic decomposition* [3]. In this paper we are going to apply this approach to a hyperbolic LPDO of order 2 with polynomial coefficients.

## 2 Hyperbolic LPDO of order 2

A bivariate operator of second order has general form

$$A_2 = a_{20}\partial_x^2 + a_{11}\partial_x\partial_y + a_{02}\partial_y^2 + a_{10}\partial_x + a_{01}\partial_y + a_{00} \quad (1)$$

and is factorizable [1] **iff**

$$a_{00} = \mathcal{L} \left\{ \frac{\omega a_{10} + a_{01} - \mathcal{L}(2a_{20}\omega + a_{11})}{2a_{20}\omega + a_{11}} \right\} + \frac{\omega a_{10} + a_{01} - \mathcal{L}(2a_{20}\omega + a_{11})}{2a_{20}\omega + a_{11}} \times \\ \times \frac{a_{20}(a_{01} - \mathcal{L}(a_{20}\omega + a_{11})) + (a_{20}\omega + a_{11})(a_{10} - \mathcal{L}a_{20})}{2a_{20}\omega + a_{11}}. \quad (2)$$

Here coefficients  $a_{i,j} = a_{i,j}(x, y)$  are functions on two variables  $x$  and  $y$ ;  $\omega$  is a distinct root of the following polynomial  $\mathcal{P}_2(z) := a_{20}z^2 + a_{11}z + a_{02}$ ,  $\mathcal{P}_2(\omega) = 0$ ; and  $\mathcal{L}$  is a linear differential operator of the form  $\mathcal{L} = \partial_x - \omega \partial_y$ .

Let us introduce a function of two variables  $x, y$

$$\mathcal{R} = \mathcal{L} \left\{ \frac{\omega a_{10} + a_{01} - \mathcal{L}(2a_{20}\omega + a_{11})}{2a_{20}\omega + a_{11}} \right\} + \frac{\omega a_{10} + a_{01} - \mathcal{L}(2a_{20}\omega + a_{11})}{2a_{20}\omega + a_{11}} \times \\ \times \frac{a_{20}(a_{01} - \mathcal{L}(a_{20}\omega + a_{11})) + (a_{20}\omega + a_{11})(a_{10} - \mathcal{L}a_{20})}{2a_{20}\omega + a_{11}},$$

and rewrite factorization condition (2) as  $a_{00} = \mathcal{R}$ .

Now suppose that (1) is a hyperbolic operator in canonical form, i.e.

$$H_2 = \partial_x^2 - \partial_y^2 + a_{10}\partial_x + a_{01}\partial_y + a_{00} \quad (3)$$

which corresponds to  $a_{20}(x, y) = 1$ ,  $a_{11}(x, y) = 0$ ,  $a_{02}(x, y) = -1$ . In this case we have two roots  $\omega_1 = 1$  and  $\omega_2 = -1$ , and function  $\mathcal{R}$  takes a form

$$\mathcal{R} = \mathcal{L} \left\{ \frac{a_{10} \pm a_{01}}{2} \right\} + \frac{(a_{10} \pm a_{01})^2}{4},$$

where "+" corresponds to  $\omega_1 = 1$  and "-" corresponds to  $\omega_2 = -1$ . We rewrite (4) in a slightly different form which will be more convenient for further use:

$$\mathcal{R} = \mathcal{L} \{S\} + S^2 \quad \text{with} \quad S = \begin{cases} (a_{10} + a_{01})/2, & \omega = 1; \\ (a_{10} - a_{01})/2, & \omega = -1. \end{cases} \quad (4)$$

### 3 Polynomial coefficients

Let us suppose that operator  $H_2$  has polynomial coefficients  $a_{ij}$  and regard cases.

#### 3.1 Polynomials of first degree

We have 3 polynomials  $a_{ij}$  of first degree with two variables  $x, y$ :  $a_{00}(x, y) = b_3x + b_2y + b_1$ ,  $a_{10}(x, y) = c_3x + c_2y + c_1$ ,  $a_{01}(x, y) = d_3x + d_2y + d_1$ , then

(1) For the first root  $\omega_1 = 1$  we have  $\mathcal{L} = \partial_x - \partial_y$  and

$$a_{10} + a_{01} = (c_3 + d_3)x + (c_2 + d_2)y + (c_1 + d_1) = f_3x + f_2y + f_1$$

with  $f_1 = (c_1 + d_1)$ ,  $f_2 = (c_2 + d_2)$ ,  $f_3 = (c_3 + d_3)$ . Then  $\mathcal{L}(a_{10} + a_{01}) = f_3 - f_2$  and

$$\mathcal{R}_1^{(1)} = \frac{f_3 - f_2}{2} + \frac{(f_3x + f_2y + f_1)^2}{4} \quad (5)$$

(2) For the second root  $\omega_2 = -1$  we have  $\mathcal{L} = \partial_x + \partial_y$  and

$$a_{10} - a_{01} = (c_3 - d_3)x + (c_2 - d_2)y + (c_1 - d_1) = h_3x + h_2y + h_1$$

with  $h_1 = (c_1 - d_1)$ ,  $h_2 = (c_2 - d_2)$ ,  $h_3 = (c_3 - d_3)$ . Then  $\mathcal{L}(a_{10} - a_{01}) = h_3 - h_2$  and

$$\mathcal{R}_2^{(1)} = \frac{h_3 - h_2}{2} + \frac{(h_3x + h_2y + h_1)^2}{4}. \quad (6)$$

**Remark 1.** Notice that functions  $\mathcal{R}_1$  and  $\mathcal{R}_2$  coincide symbolically:

$$\mathcal{R}^{(1)} = \frac{s_3 - s_2}{2} + \frac{(s_3x + s_2y + s_1)^2}{4} \quad (7)$$

but of course, the form of  $s_i$  as functions of coefficients  $c_j$ ,  $b_j$ ,  $d_j$  will be different.

**Remark 2.** Factorization condition for the operator (3) has now very simple form

$$\mathcal{R}^{(1)} = a_{00}(x, y) \Rightarrow \frac{s_3 - s_2}{2} + \frac{(s_3x + s_2y + s_1)^2}{4} = b_3x + b_2y + b_1$$

which yields

$$\begin{cases} s_3 = 0, & s_2s_3 = 0, & s_2 = 0, & s_3s_1 = 2b_3, \\ s_2s_1 = 2b_2, & s_1^2 + 2(s_3 - s_2) = 4b_1 \end{cases}$$

For instance, in case of the second root this system of equations has form

$$\begin{cases} (c_3 - d_3)^2 = 0, & (c_3 - d_3)(c_2 - d_2) = 0, & (c_2 - d_2)^2 = 0, \\ (c_3 - d_3)(c_1 - d_1) = 2b_3, & (c_2 - d_2)(c_1 - d_1) = 2b_2, \\ (c_1 - d_1)^2 + (2(c_3 - d_3) - (c_2 - d_2)) = 4b_1 \end{cases}$$

and its solution gives **all exactly factorizable operators** of this type:

$$\begin{cases} a_{00}(x, y) = (c_1 - d_1)^2/4 \\ a_{10}(x, y) = c_3x + c_2y + c_1 \\ a_{01}(x, y) = c_3x + c_2y + d_1 \end{cases}$$

### 3.2 Polynomials of second degree

Now we have 3 polynomials  $a_{ij}$  of second degree with two variables  $x, y$ :

$$a_{00}(x, y) = b_6x^2 + b_5xy + b_4y^2 + b_3x + b_2y + b_1,$$

$$a_{10}(x, y) = c_6x^2 + c_5xy + c_4y^2 + c_3x + c_2y + c_1,$$

$$a_{01}(x, y) = d_6x^2 + d_5xy + d_4y^2 + d_3x + d_2y + d_1,$$

then

1. For the first root  $\omega_1 = 1$  we have  $\mathcal{L} = \partial_x - \partial_y$  and

$$\begin{aligned} a_{10} + a_{01} &= (c_6 + d_6)x^2 + (c_5 + d_5)xy + (c_4 + d_4)y^2 + (c_3 + d_3)x + (c_2 + d_2)y + (c_1 + d_1) \\ &= f_6x^2 + f_5xy + f_4y^2 + f_3x + f_2y + f_1 \end{aligned}$$

with  $f_i = (c_i + d_i)$ ,  $\forall i = 1, 2, \dots, 6$ . Then

$$\mathcal{L}(a_{10} + a_{01}) = 2(f_6x - f_4y) + f_5(y - x) + f_3 - f_2$$

and

$$\begin{aligned} \mathcal{R}_1^{(2)} &= \frac{2(f_6x - f_4y) + f_5(y - x) + f_3 - f_2}{2} + \\ &+ \frac{(f_6x^2 + f_5xy + f_4y^2 + f_3x + f_2y + f_1)^2}{4} \end{aligned} \quad (8)$$

2. For the second root  $\omega_2 = -1$  we have  $\mathcal{L} = \partial_x + \partial_y$  and

$$\begin{aligned} a_{10} - a_{01} &= (c_6 - d_6)x^2 + (c_5 - d_5)xy + (c_4 - d_4)y^2 + (c_3 - d_3)x + (c_2 - d_2)y + (c_1 - d_1) \\ &= h_6x^2 + h_5xy + h_4y^2 + h_3x + h_2y + h_1 \end{aligned}$$

with  $h_i = (c_i - d_i)$ ,  $\forall i = 1, 2, \dots, 6$ . Then

$$\mathcal{L}(a_{10} - a_{01}) = 2(h_6x - h_4y) + h_5(y - x) + h_3 - h_2$$

and

$$\begin{aligned} \mathcal{R}_2^{(2)} &= \frac{2(h_6x - h_4y) + h_5(y - x) + h_3 - h_2}{2} + \\ &+ \frac{(h_6x^2 + h_5xy + h_4y^2 + h_3x + h_2y + h_1)^2}{4}. \end{aligned} \quad (9)$$

As above we have in fact one function

$$\begin{aligned} \mathcal{R}^{(2)} &= \frac{2(s_6x - s_4y) + s_5(y - x) + s_3 - s_2}{2} + \\ &+ \frac{(s_6x^2 + s_5xy + s_4y^2 + s_3x + s_2y + s_1)^2}{4}. \end{aligned} \quad (10)$$

**Remark 3.** As direct corollaries of linear differentiation of polynomials one can conclude that for a polynomial any finite degree  $n$ , function  $\mathcal{R}^{(n)}$  has form

$$\mathcal{R}^{(n)} = \sum_{k=1}^{n-1} \mathcal{R}^{(k)} + (\text{other terms})$$

and is a polynomial of degree  $r$  with  $r \leq 2n$ , with necessary condition of exact factorization being  $\deg(a_{00}) \leq \deg(\mathcal{R}^{(n)})$ .

## 4 Problem setting for QE

We use standard formal language of elementary real algebra, that is, Tarski algebra [3] and formulate a first simple case of approximate factorization of LPDO as a quantifier elimination (QE) problem. Namely, we shall consider the case of a hyperbolic LPDO of order 2 with polynomial coefficients of the first order. In this case we have

(1) three polynomials  $a_{ij}(x, y)$  of first degree in the two variables  $x, y$ ; the coefficients of these polynomials are also variables:

$$a_{00}(x, y) = b_3x + b_2y + b_1$$

$$a_{10}(x, y) = c_3x + c_2y + c_1$$

$$a_{01}(x, y) = d_3x + d_2y + d_1;$$

(2) one function

$$\mathcal{R}^{(1)}(x, y) = \frac{s_3 - s_2}{2} + \frac{(s_3x + s_2y + s_1)^2}{4}$$

with  $s_i$  given by (5) or by (6), i.e.  $s_i = c_i + d_i$  for the first root and  $s_i = c_i - d_i$  for the second root;

(3) a constant  $\varepsilon$ ;

(4) constants  $m$  and  $n$ , which define a bounded rectangular region in the plane:  $-m < x < m$ ,  $-n < y < n$ .

**Remark 4.** Notice that the special form of the factorization condition allowed us to reduce the number of variables needed for this QE problem. Initially we had 9 variables  $b_3, b_2, b_1, c_3, c_2, c_1, d_3, d_2, d_1$ , but in fact it is enough to consider only the 6 variables  $s_1, s_2, s_3, b_1, b_2, b_3$ .

With all this given, let us consider the quantified formula of elementary real algebra  $\phi^* = \phi^*(b_i, s_j)$  which asserts that “for all  $x$  and  $y$  in the bounded region  $-m < x < m$ ,  $-n < y < n$ , we have  $-\varepsilon < a_{00}(x, y) - \mathcal{R}^{(1)}(x, y) < \varepsilon$ .” We wish to eliminate the quantifiers from  $\phi^*(b_i, s_j)$ . More precisely, we wish to find a formula of elementary real algebra  $\phi' = \phi'(b_i, s_j)$ , free of quantifiers, such that if  $\phi'(b_i, s_j)$  is true then  $\phi^*(b_i, s_j)$  is true. That is, we wish to find

conditions on the coefficients of the initial polynomials  $a_{ij}(x, y)$  which imply that the function  $\mathcal{R}^{(1)}(x, y)$  differs not too much from one these polynomials, namely  $a_{00}(x, y)$ , throughout the bounded region  $-m < x < m$ ,  $-n < y < n$ .

## 5 Synopsis of QE by CAD

Let  $A$  be a set of integral polynomials in  $x_1, x_2, \dots, x_r$ , where  $r \geq 1$ . An *A-invariant cylindrical algebraic decomposition (CAD)* of  $\mathbf{R}^r$ ,  $r$ -dimensional real space, is a decomposition  $D$  of  $\mathbf{R}^r$  into nonempty connected subsets called cells such that

1. the cells of  $D$  are cylindrically arranged with respect to the variables  $x_1, x_2, \dots, x_r$ ;
2. every cell of  $D$  is a semialgebraic set (that is, a set defined by means of boolean combinations of polynomial equations and inequalities); and
3. every polynomial in  $A$  is sign-invariant throughout each cell of  $D$ .

The CAD algorithm as originally conceived [3, 4] has inputs and outputs as follows. Given such a set  $A$  of  $r$ -variate polynomials and a nonnegative integer  $f$  with  $f < r$ , the algorithm produces as its output a description of an  $A$ -invariant CAD  $D$  of  $\mathbf{R}^r$ , in which explicit semialgebraic defining formulas are provided only for the cells of the CAD  $D_f$  of  $\mathbf{R}^f$  induced (that is, implicitly determined) by  $D$ . The description of  $D$  comprises lists of indices and sample points for the cells of  $D$ . (Every cell is assigned an *index* which indicates its position within the cylindrical structure of  $D$ .)

The working of the original CAD algorithm can be summarized as follows. If  $r = 1$ , an  $A$ -invariant CAD of  $\mathbf{R}^1$  is constructed directly, using polynomial real root isolation. If  $r > 1$ , then the algorithm computes a *projection set*  $P$  of  $(r - 1)$ -variate polynomials (in  $x_1, \dots, x_{r-1}$ ) such that any  $P$ -invariant CAD  $D'$  of  $\mathbf{R}^{r-1}$  can be extended to a CAD  $D$  of  $\mathbf{R}^r$ . If  $f = r$  we set  $f' \leftarrow f - 1$  and otherwise set  $f' \leftarrow f$ . Then the algorithm calls itself recursively on  $P$  and  $f'$  to get such a  $D'$ . Finally  $D'$  is extended to  $D$ . In order to produce semialgebraic defining formulas for the cells of  $D_f$  the algorithm must be used in a mode called *augmented projection*.

Thus for  $r > 1$ , if we trace the algorithm, we see that it computes a first projection set  $P$ , eliminating  $x_r$ , then computes the projection of  $P$ , eliminating  $x_{r-1}$ , and so on, until the  $(r - 1)$ -st projection set has been obtained, which is a set of polynomials in the variable  $x_1$  only. This is called the *projection phase* of the algorithm. The construction of a CAD of  $\mathbf{R}^1$  invariant with respect to the  $(r - 1)$ -st projection set is called the *base phase*. The successive extensions of the CAD of  $\mathbf{R}^1$  to a CAD of  $\mathbf{R}^2$ , the CAD of  $\mathbf{R}^2$  to a CAD of  $\mathbf{R}^3$ , and so on, until an  $A$ -invariant cad of  $\mathbf{R}^r$  is obtained, constitute the *extension phase* of the algorithm.

Now we consider the *quantifier elimination (QE) problem* for the elemen-

tary theory of the reals: given a quantified formula (known as a *QE problem instance*) of elementary real algebra

$$\phi^* = (Q_{f+1}x_{f+1}) \dots (Q_r x_r) \phi(x_1, \dots, x_r)$$

where  $\phi$  is a formula involving the variables  $x_1, x_2, \dots, x_r$  which is free of quantifiers, find a formula  $\phi'(x_1, \dots, x_f)$ , free of quantifiers, such that  $\phi'$  is equivalent to  $\phi^*$ . The QE problem can be solved by constructing a certain CAD of  $\mathbf{R}^r$ . The method is described as follows.

1. Extract from  $\phi$  the list  $A$  of distinct non-zero  $r$ -variate polynomials occurring in  $\phi$ .
2. Construct lists  $S$  and  $I$  of sample points and cell indices, respectively, for an  $A$ -invariant CAD  $D$  of  $\mathbf{R}^r$ , together with a list  $F$  of semialgebraic defining formulas for the cells of the CAD  $D_f$  of  $\mathbf{R}^f$  induced by  $D$ .
3. Using  $S$ , evaluate the truth value of  $\phi^*$  in each cell of  $D_f$ . (By construction of  $D$ , the truth value of  $\phi^*$  is constant throughout each cell  $c$  of  $D_f$ , hence can be determined by evaluating  $\phi^*$  at the sample point of  $c$ .)
4. Construct  $\phi'(x_1, \dots, x_f)$  as the disjunction of the semialgebraic defining formulas of those cells of  $D_f$  for which the value of  $\phi^*$  has been determined to be true.

The above algorithm solves any given particular instance of the QE problem in principle. However the computing time of the algorithm grows steeply as the number  $r$  of variables occurring in the input formula  $\phi$  increases.

Collins and Hong [7] introduced the method of *partial CAD construction* for QE. This method, named with the acronym QEPCAD, is based upon the simple observation that we can often solve a QE problem by means of a partially built CAD. The QEPCAD algorithm was originally implemented by Hong. A recent implementation, denoted by QEPCAD-B, contains improvements by Brown, Collins, McCallum, and others – see [5]. QEPCAD-B has solved a range of reasonably interesting problems for which the original QE algorithm takes too much time. Nevertheless the worst case computing time of QEPCAD-B remains large (that is, it depends doubly-exponentially on  $r$ ).

## 6 Application of QEPCAD to BK-factorization

We consider only the first simple case of approximate factorization described in Section 4. Using the notation of Section 4, we suppose that  $\varepsilon$ ,  $m$  and  $n$  have been given specific constant values, say  $\varepsilon = m = n = 1$ , and we consider the formula  $\phi^*(b_i, s_j)$  which asserts that

$$(\forall x)(\forall y)[(|x| < 1 \wedge |y| < 1) \Rightarrow |a_{00}(x, y) - \mathcal{R}^{(1)}(x, y)| < 1]. \quad (11)$$

We wish to find a formula  $\phi'(b_i, s_j)$ , free of quantifiers, such that  $\phi'(b_i, s_j)$  implies  $\phi^*(b_i, s_j)$ .

**Remark 5.** It would be of greatest interest to find the most general such  $\phi'(b_i, s_j)$  – that is, to find quantifier-free  $\phi'(b_i, s_j)$  *equivalent* to  $\phi^*(b_i, s_j)$ . But as we'll see it seems that the time and space resources needed to do this are prohibitive. We'll also see that it is not as time consuming, yet hopefully still of interest, to find quantifier-free conditions merely *sufficient* for  $\phi^*$  to be true.

We attempted to find a solution to the above QE problem instance by running the program QEPCAD-B with the quantified formula 11 (rewritten so that the variables  $b_i, s_j$  appear explicitly, and the denominator 4 is cleared from the right hand side of the implication, see 12 below) as its input. The variable ordering used was  $(s_3, s_2, s_1, b_3, b_2, b_1, x, y)$ . The computer used for this and subsequent experiments was a Sun server having a 292 MHz ultraSPARC risc processor. Forty megabytes of memory were made available for list processing. However the program ran out of memory after approximately one hour and forty minutes. The program was executing the projection phase of the algorithm when it stopped. The first three projection steps – that is, successive elimination of  $y, x$  and  $b_1$  – were complete.

Increasing the amount of memory to eighty megabytes did not help – the program still ran out of memory during the fourth projection step (that is, during elimination of  $b_2$ ).

## 6.1 Searching for quantifier-free sufficient conditions

Of course a very special, but completely trivial, quantifier-free sufficient condition for our QE problem instance is the formula

$$\phi'(b_i, s_j) := [b_1 = 0 \wedge b_2 = 0 \wedge b_3 = 0 \wedge s_1 = 0 \wedge s_2 = 0 \wedge s_3 = 0].$$

It could be of some interest to look for partial solutions to (that is, quantifier-free sufficient conditions for) our QE problem instance in which some but not all of the variables  $b_i, s_j$  are equal to zero. For example, recall that the given quantified (11) – after rewriting so that the variables  $b_i, s_j$  appear explicitly and the denominator 4 is cleared from the right hand side of the implication – is:

$$(\forall x)(\forall y)[(|x| < 1 \wedge |y| < 1) \Rightarrow |4b_3x + 4b_2y + 4b_1 - 2(s_3 - s_2) - (s_3x + s_2y + s_1)^2| < 4]. \quad (12)$$

Suppose that we put  $b_2 = s_2 = 0$  in (12). We obtain:

$$(\forall x)(\forall y)[(|x| < 1 \wedge |y| < 1) \Rightarrow |4b_3x + 4b_1 - 2s_3 - (s_3x + s_1)^2| < 4]$$



which is equivalent to:

$$(\forall x)[(|x| < 1) \Rightarrow |4b_3x + 4b_1 - 2s_3 - (s_3x + s_1)^2| < 4], \quad (13)$$

which we shall denote by  $\psi^*(b_i, s_j)$ .

The following theorem shows that a partial solution to the special QE problem instance  $\psi^*(b_i, s_j)$  (that is, a quantifier-free sufficient condition for  $\psi^*$ ) leads to a partial solution to the QE problem instance  $\phi^*$  (that is, a quantifier-free sufficient condition for  $\phi^*$ ).

**Theorem 1** *Suppose that  $\psi'(b_i, s_j)$  is a quantifier-free formula, involving only  $b_1, b_3, s_1, s_3$ , which implies  $\psi^*(b_i, s_j)$ . Then the quantifier-free formula  $\psi'(b_i, s_j) \wedge b_2 = 0 \wedge s_2 = 0$  implies  $\phi^*(b_i, s_j)$ .*

► Let  $b_i, s_j$  be real numbers. Assume  $\psi'(b_i, s_j) \wedge b_2 = 0 \wedge s_2 = 0$ . Then  $\psi^*(b_i, s_j) \wedge b_2 = 0 \wedge s_2 = 0$  is true, by hypothesis. Take real numbers  $x$  and  $y$ , with  $|x| < 1$  and  $|y| < 1$ . Then

$$|4b_3x + 4b_2y + 4b_1 - 2(s_3 - s_2) - (s_3x + s_2y + s_1)^2| = |4b_3x + 4b_1 - 2s_3 - (s_3x + s_1)^2| < 4,$$

by virtue of (13) (since  $|x| < 1$ ). Hence (12) is true. ■

The above discussion suggests that it would be worthwhile to try to find a solution to the simplified, special QE problem instance  $\psi^*$  using the program QEPCAD-B. Putting (13) into a slightly more general form, and hence reducing by 1 the number of variables in the formula, we obtain:

$$(\forall x)[(|x| < 1) \Rightarrow |ax^2 + bx + c| < 4]. \quad (14)$$

A partial solution  $\theta'(a, b, c)$  to (14) could easily be transformed into a partial solution  $\psi'(b_1, b_3, s_1, s_3)$  to  $\psi^*$  by setting  $a = -s_3^2$ ,  $b = 4b_3 - 2s_1s_3$  and  $c = 4b_1 - 2s_3 - s_1^2$ .

We ran program QEPCAD-B with (14) as its input. Eighty megabytes of memory were made available for list processing. After 191 seconds the program produced the following quantifier-free formula equivalent to (14):

$$\begin{aligned} & c - b + a + 4 \geq 0 \wedge c - b + a - 4 \leq 0 \wedge \\ & c + b + a + 4 \geq 0 \wedge c + b + a - 4 \leq 0 \wedge \\ & [4ac - b^2 + 16a > 0 \vee 4ac - b^2 - 16a > 0 \vee \\ & [b^2 - 16a = 0 \wedge b^2 + 16a > 0] \vee [b^2 - 16a < 0 \wedge b - 2a \geq 0] \vee \\ & [b^2 - 16a < 0 \wedge b + 2a \leq 0] \vee [b^2 - 16a > 0 \wedge b + 2a \geq 0] \vee \\ & [b^2 - 16a > 0 \wedge b - 2a \leq 0] \vee \\ & [b^2 - 16a = 0 \wedge c - b + a + 4 > 0 \wedge c - b + a - 4 < 0] ]. \end{aligned}$$

Since  $a = -s_3^2$ , we have  $a \leq 0$ . We ran QEPCAD-B a second time, this time using the command

`assume [a <= 0].`

After 60 seconds the program produced the following somewhat simpler quantifier-free formula equivalent to (14) under the assumption  $a \leq 0$ :

$$\begin{aligned} & c - b + a + 4 \geq 0 \wedge c - b + a - 4 \leq 0 \wedge \\ & c + b + a + 4 \geq 0 \wedge c + b + a - 4 \leq 0 \wedge \\ & [4ac - b^2 - 16a > 0 \wedge [b > 0 \wedge b + 2a \geq 0] \wedge \\ & [b < 0 \wedge b - 2a \leq 0] \wedge \\ & [b^2 + 16a = 0 \wedge c - b + a + 4 > 0 \wedge c - b + a - 4 < 0]] . \end{aligned}$$

It is possible to induce the program to produce an arguably even simpler solution formula using less computing time by making two separate runs of QEPCAD-B. The first run uses the command

`assume [a < 0].`

After just 1.9 seconds the program produced the following quantifier-free formula equivalent to (14) under the assumption  $a < 0$ :

$$\begin{aligned} & c - b + a + 4 \geq 0 \wedge c - b + a - 4 \leq 0 \wedge \\ & c + b + a + 4 \geq 0 \wedge c + b + a - 4 \leq 0 \wedge \\ & [b - 2a \leq 0 \wedge b + 2a \geq 0 \wedge 4ac - b^2 - 16a > 0] . \quad (15) \end{aligned}$$

The above formula is perhaps the most elegant and understandable of those obtained by applying QEPCAD-B to Formula 14. For it is a slight improvement of (that is, slightly more compact than) a formula seen to be equivalent to it (under assumption  $a < 0$ ) which is quite straightforward to derive by hand from (14) using elementary properties of the parabola  $y = ax^2 + bx + c$  on the interval  $(-1, +1)$ :

$$\begin{aligned} & [2a - b \geq 0 \wedge a + b + c + 4 \geq 0 \wedge a - b + c - 4 \leq 0] \wedge \\ & [2a + b \geq 0 \wedge a - b + c + 4 \geq 0 \wedge a + b + c - 4 \leq 0] \wedge \\ & [2a - b < 0 \wedge 2a + b < 0 \wedge 4ac - b^2 - 16a > 0 \wedge \\ & a - b + c + 4 \geq 0 \wedge a + b + c + 4 \geq 0] . \quad (16) \end{aligned}$$

**Remark 6.** To derive by hand (16) from (14) under the assumption  $a < 0$ , one has to notice that function  $f(x) = ax^2 + bx + c$  has its maximum value for  $f'(x) = 2ax + b = 0$ , that is, for  $x = -b/(2a)$ , and consider three cases separately: (1)  $-b/(2a) \leq -1$ , (2)  $-b/(2a) \geq +1$ , and (3)  $-1 < -b/(2a) < +1$ . For each of the above three cases one can then write down necessary and sufficient conditions for (14) to be true. For example, in Case 1, (14) is clearly equivalent to  $-4 \leq a + b + c \wedge a - b + c \leq 4$ . After treating each of the above cases, we obtain (16) by forming the disjunction of the formulas corresponding to the cases.

Of course, (15) for  $a < 0$  is not quite a complete solution to the QE problem instance of (14) under assumption  $a \leq 0$ . To obtain a complete

solution we still needed to run QEPCAD a second time, this time for the case  $a = 0$ . For the second run we put  $a = 0$  in (14) and use the command `assume [b /= 0]`.

After 60 milliseconds the program produced the following formula equivalent to (14) with  $a = 0$  under assumption  $b \neq 0$ :

$$\begin{aligned} c - b + 4 &\geq 0 \wedge c - b - 4 \leq 0 \wedge \\ c + b + 4 &\geq 0 \wedge c + b - 4 \leq 0 \end{aligned} \quad (17)$$

This is immediately seen to be correct! Finally we could obtain a complete solution to (14) for  $a \leq 0$  by combining (15) for  $a < 0$ , (17) for  $b \neq a = 0$  and the formula  $c - 4 < 0 \wedge c + 4 > 0$  (for  $a = b = 0$ ). In fact a simple and elegant way to achieve such a combination is to insert the disjunct  $a = 0$  into the last conjunct of (15):

$$\begin{aligned} c - b + a + 4 &\geq 0 \wedge c - b + a - 4 \leq 0 \wedge \\ c + b + a + 4 &\geq 0 \wedge c + b + a - 4 \leq 0 \wedge \\ [b - 2a &\leq 0 \vee b + 2a \geq 0 \vee \\ 4ac - b^2 - 16a &> 0 \vee a = 0] \end{aligned} \quad (18)$$

## 7 Discussion

As we remarked in Section 5 the worst case computing time of QEPCAD-B grows steeply as the number of variables in the given QE problem instance increases. Indeed, as is suggested by the results reported in Section 6, a complete solution of the QE problem instance (12) by QEPCAD-B using a reasonable amount of time and space seems to be unlikely for the foreseeable future.

Nevertheless the results of Section 6 also suggest that QEPCAD-B could be of help in searching for certain kinds of sufficient conditions for (11), especially those which involve setting some of the variables to zero.

We briefly mention here another kind of approach which a person could use to derive another kind of sufficient condition for (11) by hand. Namely, one could begin by expanding the polynomial  $a_{00}(x, y) - \mathcal{R}^1(x, y)$  in terms of  $x$  and  $y$ :

$$\begin{aligned} a_{00}(x, y) - \mathcal{R}^1(x, y) &= (-s_3^2/4)x^2 + (-2s_3s_2/4)xy + (-s_2^2/4)y^2 + \\ &\quad (b_3 - (s_1s_3)/2)x + (b_2 - (s_1s_2)/2)y + (b_1 - (s_3 - s_2)/2 - s_1^2/4). \end{aligned}$$

By inspection of the terms on the right hand side of the above equation we see that a sufficient condition for (11) is:

$$\begin{aligned} |s_3^2/4| &< 1/6 \wedge |2s_3s_2/4| < 1/6 \wedge \\ |s_2^2/4| &< 1/6 \wedge |b_3 - (s_1s_3)/2| < 1/6 \wedge \\ |b_2 - (s_1s_2)/2| &< 1/6 \wedge \\ |b_1 - (s_3 - s_2)/2 - s_1^2/4| &< 1/6. \end{aligned}$$

The above sufficient condition is unlikely to be obtained in a reasonable amount of time and space using QEPCAD-B applied to (12), even if one issues **assume** commands. The number of variables involved is probably too big. However a version of QEPCAD-B which is planned for the future, which will have the capability to determine adjacency relationships amongst the cells of the partial CAD, could be of some use in analyzing certain topological properties of the truth set in six-dimensional space of the quantifier-free formula in  $b_i, s_j$  above.

## Acknowledgements

This paper has its origin in discussions between the authors at RISC-Linz during the second of S.M. to RISC-Linz in 2005. S.M. would like to thank Professors Franz Winkler and Bruno Buchberger, and all their colleagues and staff at RISC, for their hospitality during his stay. S.M. would also like to acknowledge helpful discussions and communications with Professors Daniel Lazard and Chris Brown. E.K. acknowledges the support of the Austrian Science Foundation (FWF) under projects SFB F013/F1304.

## References

- [1] R. Beals, E. Kartashova. "Constructively factoring linear partial differential operators in two variables." *TMPh* **145**(2): 1510-1523 (2005)
- [2] E. Kartashova, O. Rudenko. "Invariant Form of BK-factorization and its Applications." *Proc. GIFT-2006*: 225-241. Eds.: J. Calmet, W.M. Seiler, R.W. Tucker. Universitätsverlag Karlsruhe (2006)
- [3] G. E. Collins. "Quantifier elimination for real closed fields by cylindrical algebraic decomposition." **LNCS 33**: 134-183. Springer Verlag, Berlin (1975)
- [4] D. Arnon, G. Collins, S. McCallum. "Cylindrical algebraic decomposition I: the basic algorithm." *JSC*, **13**(4): 878-889, (1984)
- [5] C. Brown. "QEPCAD B: a program for computing with semi-algebraic sets using CADs." *ACM SIGSAM Bulletin*, **37**(4): 97-108, (2003).
- [6] B. F. Caviness, J. R. Johnson (Eds.) **Quantifier Elimination and Cylindrical Algebraic Decomposition**. Springer Verlag, Berlin (1998)
- [7] G. E. Collins, H. Hong. "Partial cylindrical algebraic decomposition for quantifier elimination." *JSC*, **12**(3): 299-328, (1991)